PHYS 681/682, Lecture Notes on Quantum Field Theory

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1 Non-Abelian Gauge Theory, Srednicki §24 & §69

1.1 Compact Lie Group SU(N)

Suppose field $\phi(x)$ has N components $\phi_i(x), i = 1, 2, \cdots N$, representing some internal degrees of freedom.

$$SU(N) = \left\{U \in GL(N, \mathbb{C}) : \det U = 1, U^\dagger U = 1\right\}$$

is the collection of $N \times N$ special unitary matrices (transformations). The transform reads,

$$\phi_i(x) \to \phi'_i(x) = U_{ij}\phi_j(x), \tag{1}$$

Each transformation U can be written in exponential form,

$$U = e^{-i\alpha^a\tau^a}, a = 1, 2, \cdots, N^2 - 1 \tag{2}$$

where $\alpha^a \in \mathbb{N}$, τ^a are $N \times N$ traceless hermitian matrices, called generators of SU(N). Hermicity comes from unitarity of U; tracelessness is a result of $\det U = 1$ by the identity $\det e^A = e^{\mathrm{tr}A}$. All $N \times N$ traceless hermitian matrices form a real vector space $\mathfrak{su}(N)$. The dimensionality of $\mathfrak{su}(N)$ is $N^2 - 1$. That's why we have $N^2 - 1$ generators. Any traceless hermitian matrix can be written as a superposition of τ^a's.

$$M = \sum_a C_a\tau^a$$

The coefficients C_a are real numbers.

1.2 Lie Algebra $\mathfrak{su}(N)$

SU(N) group also defines a commutation relation for τ^a. Consider infinitesmal transformations, $U = 1 - i\epsilon\alpha^a\tau^a + \mathcal{O}(\epsilon^2), V = 1 - i\epsilon\beta^a\tau^a + \mathcal{O}(\epsilon^2)$. $W = U^{-1}V^{-1}UV \in SU(N)$ due to the closeness of a group. It can be seen, $W = 1 - \epsilon^2\alpha^a\beta^b\left[\tau^a, \tau^b\right] + \mathcal{O}(\epsilon^3)$. W is also an infinitesmal transformation, fully determined by group multiplication rules. Suppose $W = e^{-i\epsilon^2F^c(\alpha^a,\beta^b)\tau^c}$. Clearly, $\left[\tau^a, \tau^b\right]$ is fully determined by function $F^c(\alpha^a, \beta^b)$.

It's easy to see, commutator $i\left[\tau^a, \tau^b\right]$ is hermitian and traceless: $\mathrm{tr}\, i\left[\tau^a, \tau^b\right] = i\mathrm{tr}(\tau^a\tau^b) - i\mathrm{tr}(\tau^b\tau^a) = 0$. Therefore, we can rewrite it use τ^c's again, $\left[\tau^a, \tau^b\right] = if^{abc}\tau^c$, where summation over c is implicit. The coefficients f^{abc} are real and fully determined by function F^c, the group multiplication rule. f^{abc} are called structure constants. For non-abelian groups, f^{abc} do not all vanish.

A vector space with a commutator defined on it is called a Lie algebra (Mathematians describe Lie algebra with fancier languages). Similary as we have constructed, each compact Lie group G is associated with a Lie algebra, usually denoted as $\mathfrak{g}$. So $\mathfrak{su}(N)$ is the Lie algebra of $SU(N)$ group. τ^a are again called generators. Note that $\mathfrak{su}(N)$ is a real algebra, because it's a real vector space.

As a vector space, we also define inner product over $\mathfrak{su}(N)$:

$$(a, b) = \mathrm{tr}ab, \forall a, b \in \mathfrak{su}(N)$$

With the help of inner product, we normalize the generators (the basis of the vector space),

$$\mathrm{tr}\tau^a\tau^b = \frac{1}{2}\delta^{ab} \tag{3}$$

If we can defined positive norm for all elements, i.e. $\forall a, b \in \mathfrak{g}, (a, b) \geq 0,$, then G is called a compact Lie group. $SU(N), SO(N)$ are compact groups. Other compact simple Lie groups include, $Sp(N), G(2), F(4), E(6), E(7), E(8)$. Quantum theory of gauge fields is based on compact groups. Standard model is based on $U(1) \times SU(2) \times SU(3)$. People generalize it to larger compact groups to make a grand unified theory (GUT). Georgi-Glashow model takes $SU(5)$. $SO(10)$ and $E(6)$ are another two popular models.

1.3 Gauge Symmetry

In $SU(N)$ symmetry, a transformation $U = e^{-i\alpha^a\tau^a}, \alpha^a \in \mathbb{R}$ does not depend on space-time. In $SU(N)$ gauge symmetry, this global transformation is enlarged to a local one $U(x) = e^{-i\alpha^a(x)\tau^a}$, where $\alpha^a(x) \in C^\infty(\mathbb{R})$ is some smooth real function. A gauge symmetry (gauge group) is hence also called a local symmetry (local group). Similarly $SU(N)$ symmetries are given the name global $SU(N)$ symmetries.

example of gauge theories

$U(1) \simeq SO(2)$	electromagnetism
$U(1) \times SU(2)$	electroweak interaction
$SU(3)$	quantum chromodynamics
$ISO(1, 3)$	general relativity

where, $ISO(1, 3)$ is the Poincaré group, Lorentz transformations plus translations.

Take GR as an example. Poincaré symmetry preserve the Minkowski space quadratic form,

$$\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}, \quad \text{where } g^{\mu\nu} = \mathrm{diag}\{-1, +1, +1, +1\}$$

GR enlarges this symmetry to a local one,

$$\Lambda^\mu{}_\alpha(x)\Lambda^\nu{}_\beta(x)g^{\alpha\beta}(x) = g^{\mu\nu}(x),$$

where metric $g^{\mu\nu}(x)$ is a tensor field with signature $(-1, +1, +1, +1)$. The gauged group is simply the space-time coordinate transform, the diffeomorphism group of the Riemann space.

1.4 Gauge Invariants and Gauge Covariant Derivatives

Let $V(x)$ be an $SU(N)$ gauge transformation. Under the gauge transformation, scalar fields transform

$$\phi_i(x) \to \phi'_i(x) = V_{ij}(x)\phi_j(x).$$

So $\phi_i^\dagger(x)\phi_i(x) \to \phi_j^\dagger(x)V_{ji}^\dagger(x)V_{ik}(x)\phi_k(x) = \phi_i^\dagger(x)\phi_i(x)$ is gauge invariant. Similarly, spinor fields transform as

$$\psi_i(x) \to \psi'_i(x) = V_{ij}(x)\psi_j(x).$$

$\bar{\psi}_i(x)\psi_i(x) \to \psi_i^\dagger(x)V^\dagger(x)\beta V(x)\psi_i(x) = \bar{\psi}_i(x)\psi_i(x)$ is also a gauge invariant. However, the derivative term $\partial_\mu\phi_i(x)$transforms as,

$$\partial_\mu\phi_i(x) \to \partial_\mu(V_{ij}(x)\phi_j(x)) = V_{ij}(x)\partial_\mu\phi_j(x) + \partial_\mu V_{ij}(x)\phi_j(x)$$

The quadratic derivative term,

$$\begin{aligned}
&\partial_\mu\phi_i^\dagger(x)\partial^\mu\phi_i(x) \to \\
&\partial_\mu\phi_i^\dagger(x)\partial^\mu\phi_i(x) + \phi_j^\dagger(x)\partial_\mu U_{ji}^\dagger(x)V_{ik}(x)\partial^\mu\phi_k(x) + \partial_\mu\phi_j^\dagger(x)V_{ji}^\dagger(x)\partial^\mu V_{ik}(x)\phi_k(x) + \phi_j^\dagger(x)\partial_\mu V_{ji}^\dagger(x)\partial^\mu V_{ik}(x)\phi_k(x) \\
=&\partial_\mu\phi_i^\dagger(x)\partial^\mu\phi_i(x) + \phi_j^\dagger(x)\partial_\mu V_{ji}^\dagger(x)\partial^\mu V_{ik}(x)\phi_k(x)
\end{aligned}$$

is not gauge invariant. We have used $\partial_\mu(V^\dagger(x)V(x)) = \partial_\mu 1 = 0 \implies \partial_\mu V^\dagger(x)V(x) + V^\dagger(x)\partial_\mu V(x) = 0$.

Covariant Derivatives in Curved Space (GR) Let $V^\mu(x)$ be a tangent vector field in a curved space (See Fig. 1). The derivative is defined as,

$$\xi^\alpha\partial_\alpha V^\mu(x) = \lim_{\epsilon\to 0}\frac{V^\mu(x+\epsilon\xi) - V^\mu(x)}{\epsilon} \tag{4}$$

Taking derivative is to compare vector fields at two infinitesmally close points. Since they are defined at different space-time points, direct comparison violates the principle of locality. A direct comparison of two separated vectors requires knowing their globally defined coordinates, which is not known. Another way to compare the two vectors is to parallelly transport vector $V^\mu(x)$ from x to $x+\epsilon\xi$ and then compare. Suppose the parallel transport is $T^\mu{}_\nu(y,x)$, we define the covariant direvative,

$$\xi^\alpha D_\alpha V^\mu(x) = \lim_{\epsilon\to 0}\frac{V^\mu(x+\epsilon\xi) - T^\mu{}_\nu(x+\epsilon\xi, x)V^\nu(x)}{\epsilon} \tag{5}$$

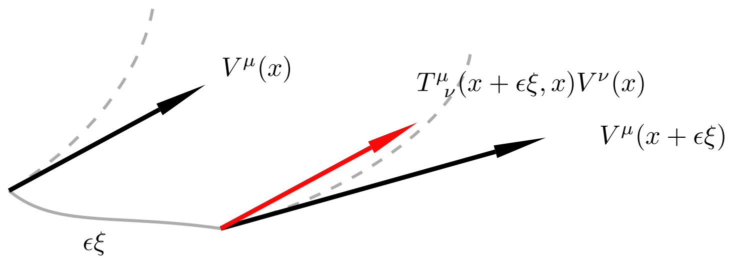

Figure 1

Covariant Derivative in Gauge Theories In gauge theories, the covariant derivative is also defined as,

$$\xi^\alpha D_\alpha\phi_i(x) = \lim_{\epsilon\to 0}\frac{\phi_i(x+\epsilon\xi) - U_{ij}(x+\epsilon\xi, x)\phi_j(x)}{\epsilon} \tag{6}$$

where $U_{ij}(y,x)$ is a parallel transport (also called comparator, gauge link) in the internal space. The parallel transport havs the following properties,

- It's path dependent: $U(y,x) = U_\gamma(y,x) \equiv U[\gamma]$, where $\gamma(s)$ is a curve, $\gamma(0) = x$, $\gamma(1) = y$;
- $U_\gamma(x,x) = 1$;
- Suppose $\gamma(0) = \sigma(1)$, $U_\gamma(z,y)U_\sigma(y,x) = U_{\gamma\circ\sigma}(z,x)$;
- $U(y,x)$ should be unitary; In non-abelian gauge theory, we further require $\det U = 1$;

- $U(y,x)$ transforms as $U(y,x) \to V(y)U(y,x)V^\dagger(x)$;

  In this way, the covariant derivative transforms covariantly:

$$\begin{aligned}\xi^\mu D_\mu\phi(x) \to &\lim_{\epsilon\to 0}\frac{V(x+\epsilon\xi)\phi(x+\epsilon\xi) - V(x+\epsilon\xi)U(x+\epsilon\xi,x)V^\dagger(x)V(x)\phi(x)}{\epsilon}\\ =&\lim_{\epsilon\to 0}V(x+\epsilon\xi)\frac{\phi(x+\epsilon\xi)-U(x+\epsilon\xi,x)\phi(x)}{\epsilon}\\ =&V(x)\xi^\mu D_\mu\phi(x)\end{aligned}$$

We define the infinitesmal parallel transport,

$$U(x+\epsilon\xi,x) \equiv 1 + ig\epsilon\xi^\mu A_\mu(x) + \mathcal{O}(\epsilon^2). \tag{7}$$

$A_\mu(x)$ is called the gauge field. It's a $N\times N$ traceless hermitian matrix following the unitarity and determinant 1 of $U(y,x)$. The transformation property of $U(y,x)$ requires $A_\mu(x)$ to transform,

$$\begin{aligned}1 + ig\epsilon\xi^\mu A_\mu(x) + \mathcal{O}(\epsilon^2) \to &V(x+\epsilon\xi)(1+ig\epsilon\xi^\mu A_\mu(x))V^\dagger(x)\\ =&\left(V(x)+\epsilon\xi^\mu\partial_\mu V(x)\right)\left(1+ig\epsilon\xi^\mu A_\mu(x)\right)V^\dagger(x)+\mathcal{O}(\epsilon^2)\\ =&1+\epsilon\xi^\mu\partial^\mu V(x)V^\dagger(x)+ig\epsilon\xi^\mu V(x)A_\mu(x)V^\dagger(x)+\mathcal{O}(\epsilon^2)\\ =&1+ig\epsilon\xi^\mu V(x)\left(A_\mu(x)+ig^{\text{-}1}\partial_\mu\right)V^\dagger(x)+\mathcal{O}(\epsilon^2)\end{aligned}$$

where we have used $\partial_\mu(V(x)V^\dagger(x)) = \partial_\mu 1 = 0 \implies \partial_\mu V(x)V^\dagger = -V(x)\partial_\mu V^\dagger(x)$. Therefore, $A_\mu(x)$ must transform as,

$$\begin{aligned}A_\mu(x) \to A'_\mu(x) =&V(x)A_\mu(x)V^\dagger(x) + ig^{\text{-}1}V(x)\partial_\mu V^\dagger(x)\\ =&V(x)A_\mu(x)V^\dagger(x) - \partial_\mu\alpha^a(x)V(x)\tau^aV^\dagger(x)\\ =&\left(A^a_\mu(x)-\partial_\mu\alpha^a(x)\right)V(x)\tau^aV^\dagger(x)\end{aligned} \tag{8}$$

where $V(x) = e^{-ig\alpha^a(x)\tau^a}$. If $V(x)$ is an infinitesmal transformation,

$$A^c_\mu(x) \to A'^c_\mu(x) = A^c_\mu(x) - \partial_\mu\alpha^c(x) - gf^{abc}A^a_\mu(x)\alpha^b(x) + \mathcal{O}(\alpha^2) \tag{9}$$

We have used Baker-Campbell-Hausdorff formula $\exp(X)Y\exp(-X) = Y + [x,Y] + \frac{1}{2}[X,[X,Y]] + \cdots$.

In abelian gauge theories $f^{abc} = 0$,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x) \tag{10}$$

But this is true even for finite transformations, because gauge transformations and gauge field are c-number functions, and can commute freely in Eq. 8.

Therefore, the covariant derivative is,

$$D_\mu = \partial_\mu - igA_\mu(x). \tag{11}$$

Now $D_\mu\phi(x) \to V(x)D_\mu\phi(x) = V(x)D_\mu\left(V^\dagger(x)V(x)\phi(x)\right)$. Since $\phi(x)\to V(x)\phi(x)$, we denote $D_\mu \to V(x)D_\mu V^\dagger(x)$ as the transform of D_μ. Note that D_μ contains derivative, so it still eats everything on its right.

Apparently, $(D_\mu\phi(x))^\dagger D^\mu\phi(x)$ and $\bar\psi(x)D_\mu\psi(x)$ are gauge invariants.

Field Tensor Consider $[D^\mu, D^\nu]\phi(x) \equiv D^\mu(D^\nu\phi(x)) - D^\nu(D^\mu\phi(x))$.

$$\begin{aligned}&[D^\mu,D^\nu]\phi(x)\\ =&[\partial^\mu,\partial^\nu]\phi(x) - ig[\partial^\mu,A^\nu]\phi(x) - ig[A^\mu,\partial^\nu]\phi(x) - g^2[A^\mu,A^\nu]\phi(x)\\ =&-ig\left(\partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu,A^\nu]\right)\phi(x)\end{aligned}$$

As we can see, although D^μ is a differential operator, $[D^\mu, D^\nu]$ is only a regular function. We denote $F^{\mu\nu} \equiv \frac{i}{g}[D^\mu,D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu,A^\nu]$. It is a also $N\times N$ hermitian traceless matrix, and can be written as, $F^{\mu\nu} \equiv F^{\mu\nu a}\tau^a$. So $F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + gf^{abc}A^{\mu b}A^{\nu c}$.

It transforms as $F^{\mu\nu} \to V(x)D^\mu V^\dagger(x)V(x)D^\nu V^\dagger(x) - V(x)D^\nu V^\dagger(x)V(x)D^\mu V^\dagger(x) = V(x)F^{\mu\nu}V^\dagger(x)$. We can construct a gauge invariant from it, $\text{tr}F^2 \equiv \text{tr}F^{\mu\nu}F_{\mu\nu}$.

Finite Parallel Transport and Wilson Loops Let $\gamma(s), s \in [0,1]$ be some path. If a field $\phi(x)$ is parallelly transported along γ, $\phi(\gamma(s)) = U(\gamma(s), \gamma(0))\phi(\gamma(0)), \forall s \in [0,1]$.

$$\begin{aligned}
\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s} D_\mu \phi(\gamma(s)) &= \frac{\phi(\gamma(s+\mathrm{d}s)) - U(\gamma(s+\mathrm{d}s), \gamma(s))\phi(\gamma(s))}{\mathrm{d}s} \\
&= \frac{U(\gamma(s+\mathrm{d}s), \gamma(0)) - U(\gamma(s+\mathrm{d}s), \gamma(s))U(\gamma(s), \gamma(0))}{\mathrm{d}s}\phi(\gamma(0)) \\
&= 0
\end{aligned} \tag{12}$$

Here $\gamma(s) = x(s)$ and $\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s} = \frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}$ is the tangential vector along the path γ. so $\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s}\partial_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\partial}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}s}$.

So we have defined an initial value problem:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\phi(x(s)) = -ig\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A_\mu(x(s))\phi(x(s)); \\
&\phi(x(0)) = \phi(\gamma(0)).
\end{aligned} \tag{13}$$

Recall initial value problem of Schrödinger's equation,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = -iH\left|\psi(t)\right\rangle; \\
&\left|\psi(0)\right\rangle = \left|\psi_0\right\rangle.
\end{aligned}$$

The solution is $|\psi(t)\rangle = \mathcal{T}\exp\left\{-i\int_0^t \mathrm{d}\tau H(\tau)\right\}|\psi_0\rangle$. Similarly, The solution of Eq. 13 is

$$\phi(\gamma(s)) = \mathcal{P}\exp\left\{-ig\int_\gamma \mathrm{d}s\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A_\mu(x(s))\right\}\phi(\gamma(0))$$

where $\mathcal{P}$ is path ordering operator. Apparently,

$$U[\gamma] = \mathcal{P}\exp\left\{-ig\int_\gamma \mathrm{d}s\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A_\mu(x(s))\right\} \equiv \mathcal{P}\exp\left\{-ig\int_\gamma \mathrm{d}x^\mu A_\mu(x)\right\} \tag{14}$$

If γ is a loop, namely $\gamma(0) = \gamma(1)$, $W_\gamma(x) \equiv \mathrm{tr}U_\gamma(x,x) = \mathcal{P}\exp\left\{-ig\oint_\gamma \mathrm{d}x^\mu A_\mu(x)\right\}$ is gauge invariant, called Wilson loop. In GR, parallel transportation along a closed loop yields the curvature tensor $R^\mu_{\ \nu\lambda\rho}$. An analog in non-abelian gauge theory is the field tensor. In abelian theory, using Stokes Theorem, Wilson loop becomes,

$$W_\gamma(x) = \exp\left\{-ig\int_S \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\ \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right\} = \exp\left\{-ig\int_S \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\ F_{\mu\nu}\right\}$$

$\partial S = \gamma$.

In non-abelian case, Stokes theorem has to be generalized (See [1] for example.). One can avoid heavy mathematics by calculating an infinitesmal loop enclosed by $[x, x+\mathrm{d}x^\mu, x+\mathrm{d}x^\mu+\mathrm{d}y^\nu, x+\mathrm{d}y^\nu]$.

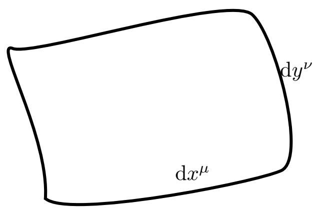

Figure 2

$$U(x+\mathrm{d}x, x) = 1 - ig\mathrm{d}x^\mu A_\mu(x+\tfrac{1}{2}\mathrm{d}x) - \tfrac{1}{2}g^2\mathrm{d}x^\mu\mathrm{d}x^\nu A_\mu(x)A_\nu(x) + \mathcal{O}(\mathrm{d}^3);$$
$$U(x+\mathrm{d}x+\mathrm{d}y, x+\mathrm{d}x) = 1 - ig\mathrm{d}y^\mu A_\mu(x+\mathrm{d}x+\tfrac{1}{2}\mathrm{d}y) - \tfrac{1}{2}g^2\mathrm{d}y^\mu\mathrm{d}y^\nu A_\mu(x+\mathrm{d}x)A_\nu(x+\mathrm{d}x) + \mathcal{O}(\mathrm{d}^3);$$

$U(x+\mathrm{d}y, x+\mathrm{d}x+\mathrm{d}y) = 1+ig\mathrm{d}x^\mu A_\mu(x+\frac{1}{2}\mathrm{d}x+\mathrm{d}y)-\frac{1}{2}g^2\mathrm{d}x^\mu\mathrm{d}x^\nu A_\mu(x+\mathrm{d}x+\mathrm{d}y)A_\nu(x+\mathrm{d}x+\mathrm{d}y)+\mathcal{O}(\mathrm{d}^3)$;
$U(x, x+\mathrm{d}y) = 1+ig\mathrm{d}y^\mu A_\mu(x+\mathrm{d}y)-\frac{1}{2}g^2\mathrm{d}y^\mu\mathrm{d}y^\nu A_\mu(x+\frac{1}{2}\mathrm{d}y)A_\nu(x+\mathrm{d}y)+\mathcal{O}(\mathrm{d}^3)$.

$$\begin{aligned}U_\gamma(x) =&U(x,x+\mathrm{d}y)U(x+\mathrm{d}y,x+\mathrm{d}x+\mathrm{d}y)U(x+\mathrm{d}x+\mathrm{d}y,x+\mathrm{d}x)U(x+\mathrm{d}x,x),\\ =&1-ig\mathrm{d}x^\mu\mathrm{d}y^\nu F_{\mu\nu}+\mathcal{O}(\mathrm{d}^3)\end{aligned}$$

Covariance of field tensor follows from the covariance of $U_\gamma(x)$. That is to say, gauge link is more fundamental than field tensor.

θ term Define the dual field tensor $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$, is gauge covariant. We can construct two quadratic gauge invariants: $\mathrm{tr}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$ and $\mathrm{tr}\tilde{F}^{\mu\nu}F_{\mu\nu}$. But $\mathrm{tr}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = \mathrm{tr}F^{\mu\nu}F_{\mu\nu}$:

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}\epsilon_{\mu\nu\sigma\kappa}F_{\rho\lambda}F^{\sigma\kappa} = \frac{1}{2}\left(\delta^\rho_{\ \sigma}\delta^\lambda_{\ \kappa}-\delta^\rho_{\ \kappa}\delta^\lambda_{\ \sigma}\right)F_{\rho\lambda}F^{\sigma\kappa} = F^{\sigma\kappa}F_{\sigma\kappa}$$

In abelian gauge theory (electromagnetism), $F^{\mu\nu}\tilde{F}_{\mu\nu} = -4E\cdot B$ is a total divergence term,

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu\left(\epsilon^{\mu\rho\kappa\lambda}F_{\rho\kappa}A_\lambda\right)$$

In non-ablian gauge theories, $F^{\mu\nu}\tilde{F}_{\mu\nu}$ is also a total divergence term, but it has a topological effect.

Yang-Mills Theory Yang-Mills theory admits a Lagrangian,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\not{D}-m\right)\psi \tag{15}$$

where $D_\mu = \partial_\mu - igA_\mu$. In terms of color components,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} + \bar{\psi}_i\left(i\not{D}_{ij}-m\delta_{ij}\right)\psi_j \tag{16}$$

where $\not{D}_{ij} = \gamma^\mu\partial_\mu\delta_{ij} - ig\gamma^\mu A^a_\mu\tau^a_{ij}$.

References

[1] N. E. Bralić, Phys. Rev. D 22 (1980) 3090